K-FLATNESS AND ORTHOGONALITY IN HOMOTOPY CATEGORIES

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Abstract. K-flatness for unbounded complexes of modules over a ring R was introduced by Spaltenstein [25], as an analogue of the classical notion of flatness for modules. In this paper, we show that the class $\mathbf{K}$-flat consisting of all K-flat complexes is precovering in the homotopy category $\mathbf{K}(R)$. In fact, a Bousfield localization exists for the embedding $\mathbf{K}\text{-flat} \subseteq \mathbf{K}(R)$ and the quotient $\mathbf{K}(R)/\mathbf{K}\text{-flat}$ is equivalent to the homotopy category of acyclic complexes of pure injective modules. When restricting to the homotopy category $\mathbf{K}(\text{Flat})$ of flat modules, we recover the fact that a Bousfield localization exists for the embedding $\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}) \subseteq \mathbf{K}(\text{Flat})$ and the existence of an equivalence between the quotient $\mathbf{K}(\text{Flat})/(\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}))$ and the homotopy category of acyclic complexes of flat cotorsion modules. The proofs use Stovicek's result [26] on the closure of the left Hom-orthogonal of certain complexes under filtered colimits.

Contents

0. Introduction

The use of unbounded complexes is often unavoidable, when dealing with mathematical problems that employ homological techniques. For example, unbounded complexes are indispensable tools in the following subjects:

(1) the study of group actions on homotopy spheres, which is closely related to the analysis of periodicity phenomena in group cohomology that can be understood in terms of the generalized Tate (complete) cohomology; cf. [2], [10], [21], [27], [28],

(2) the relative homological theory, which is based on the notion of G-dimension introduced by Auslander and Bridger [3], and was fruitfully extended by Enochs and Jenda [13] to find applications in representation theory and algebraic geometry and

(3) the approach to Grothendieck duality by Iyengar and Krause [16], Neeman [24] and Murfet [22], who showed that a dualizing complex induces a certain equivalence of triangulated categories of unbounded complexes of sheaves.

When dealing with resolutions of unbounded complexes, the proper analogue of the concepts of projective and injective modules consists of the notions introduced by Spaltenstein in [25];

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see also [4]. A complex of R-modules P is called K-projective if the complex of abelian groups $\mathrm{Hom}_R(P,C)$ is acyclic for any acyclic complex of R-modules C. Dually, a complex I is called K-injective if the complex of abelian groups $\mathrm{Hom}_R(C,I)$ is acyclic for any acyclic complex C. Finally, a complex of (left) R-modules F is called K-flat if the complex of abelian groups $L \otimes_R F$ is acyclic for any acyclic complex of right R-modules L. The conceptual similarity between projective, injective and flat modules to their K-version counterparts for complexes is supported by the following properties of K-flat complexes:

(i) The class of K-flat complexes contains the class of K-projective complexes and is closed under filtered colimits.

(ii) Any K-flat complex is homotopy equivalent to a filtered colimit of K-projective complexes; this is the analogue of the Lazard-Govorov theorem on flat modules (cf. [15], [20]).

(iii) A complex X is K-flat if and only if the Pontryagin dual complex of right R-modules DX is K-injective; this is the analogue of Lambek's flatness criterion [19].

The proof of properties (i) and (iii) is easy, whereas a proof of (ii) can be found in [12]. In this paper, we present yet another similarity between the flatness of modules and the K-flatness of complexes. In order to formulate that similarity, the relevant concepts are: cotorsion pairs in the module category R-Mod on one hand and Bousfield localizing pairs in the homotopy category $\mathbf{K}(R)$ on the other.

The class Flat of flat modules can be used in order to define the class Cotor of cotorsion modules as its right Ext^1-orthogonal; an R-module C is cotorsion if and only if $\mathrm{Ext}^1_R(F,C)=0$ for all flat modules F. The pair $(\mathrm{Flat},\mathrm{Cotor})$ has the following properties:

(a) An R-module F is flat if and only if $\mathrm{Ext}^1_R(F,C)=0$ for all cotorsion modules C.

(b) The class Flat is closed under kernels of epimorphisms.

(c) For any module M, there are short exact sequences of modules

$$0 \longrightarrow C \longrightarrow F \xrightarrow{p} M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M \longrightarrow C' \longrightarrow F' \longrightarrow 0,$$

where the modules F,F' are flat and the modules C,C' are cotorsion.

Proving properties (a) and (b) is easy, whereas property (c) follows from the proof of the flat cover conjecture given in [6]. The linear map p in the first of the two exact sequences in (c) above induces an epimorphism of abelian groups $p_* : \mathrm{Hom}_R(F_0,F) \longrightarrow \mathrm{Hom}_R(F_0,M)$ for any flat module F_0; we say that p is a (special) flat precover of M. The three properties (a), (b) and (c) entitle the pair $(\mathrm{Flat},\mathrm{Cotor})$ to be a complete and hereditary cotorsion pair in the module category R-Mod.

The analogue of the orthogonality relation between modules induced by the Ext^1-pairing is the orthogonality relation between complexes induced by the Hom-pairing in the homotopy category $\mathbf{K}(R)$. Two complexes X,Y are Hom-orthogonal to each other if $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y)=0$, i.e. if any cochain map from X to Y is null-homotopic. The subcategory $\mathbf{K}$-flat of the homotopy category $\mathbf{K}(R)$, which consists of all K-flat complexes, may be used in order to define its right Hom-orthogonal class $\mathfrak{Y}$; a complex Y is in $\mathfrak{Y}$ if and only if $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y)=0$ for all K-flat complexes X. Then, the pair $(\mathbf{K}\text{-flat},\mathfrak{Y})$ has the following properties:

(α) A complex X is K-flat if and only if $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y)=0$ for all complexes $Y \in \mathfrak{Y}$.

(β) $\mathbf{K}$-flat is a thick triangulated subcategory of the homotopy category $\mathbf{K}(R)$.

(γ) For any complex Z, there is a distinguished triangle in $\mathbf{K}(R)$

$$X \xrightarrow{f} Z \longrightarrow Y \longrightarrow SX,$$

where X is K-flat and $Y \in \mathfrak{Y}$.
Property (β) is immediate from the definition of K-flatness. In this paper, we prove properties (α), (γ) and show that the right orthogonal $\mathfrak{Y}$ consists precisely of those acyclic complexes Y, which are homotopy equivalent to a complex of pure injective modules, i.e. $\mathfrak{Y} = \mathbf{K}_{ac}(\text{PInj})$. We note that the morphism f in the distinguished triangle in (γ) above induces an isomorphism of abelian groups $f_* : \text{Hom}_{\mathbf{K}(R)}(X_0, X) \longrightarrow \text{Hom}_{\mathbf{K}(R)}(X_0, Z)$ for any K-flat complex X_0. In particular, f is a K-flat precover of Z and hence the subcategory $\mathbf{K}$-flat of $\mathbf{K}(R)$ is precovering. In fact, the isomorphism f_* shows that the object $X \in \mathbf{K}$-flat represents the restriction of the functor $\text{Hom}_{\mathbf{K}(R)}(\_, Z)$ to the full subcategory $\mathbf{K}\text{-flat} \subseteq \mathbf{K}(R)$. It follows that the inclusion $\mathbf{K}\text{-flat} \hookrightarrow \mathbf{K}(R)$ admits a right adjoint (which maps Z onto X). The three properties (α), (β) and (γ) entitle the pair $(\mathbf{K}\text{-flat}, \mathfrak{Y}) = (\mathbf{K}\text{-flat}, \mathbf{K}_{ac}(\text{PInj}))$ to be a Bousfield localizing pair in the homotopy category $\mathbf{K}(R)$; cf. [18], [23].

The class PInj of pure injective R-modules is the class of injective objects for the pure exact structure in the module category R-Mod; cf. [17], [29]. It turns out that $\text{PInj} \subseteq \text{Cotor}$ and the latter inclusion enjoys the following (density) property: For any R-module F the kernel of the functor $\text{Ext}^1_R(F, \_)$ contains all pure injective modules if and only if it contains all cotorsion modules; both assertions are equivalent to the flatness of F. If I is pure injective, then the kernel of the functor $\text{Ext}^1_R(\_, I)$ is closed under filtered colimits. Stovicek proved in [26] that an analogous property holds for complexes of pure injective modules. More precisely, he proved that for any complex of pure injective modules Y the class of those complexes X for which $\text{Hom}_{\mathbf{K}(R)}(X, S^n Y) = 0$ for all $n \in \mathbb{Z}$ is closed under filtered colimits. This is the main technical tool that we employ to prove the properties of K-flat complexes mentioned above.

In fact, we may use a slight generalization of Stovicek's result and recover some already known properties of K-flat complexes of flat modules. Let $\mathbf{K}(\text{Flat})$ be the homotopy category of flat modules and $\mathbf{K}(\text{Cotor})$ the homotopy category of cotorsion modules. Then, the classes $\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat})$ and $\mathbf{K}_{ac}(\text{Cotor})$ are orthogonal to each other, i.e. $\text{Hom}_{\mathbf{K}(R)}(X, Y) = 0$ for any K-flat complex of flat modules X and any acyclic complex of cotorsion modules Y. As a consequence, it follows that the cosyzygy modules of any acyclic complex of cotorsion modules are also cotorsion; this result has been proved by Bazzoni et al. in [5]. On the other hand, this latter result implies, in view of Gillespie's work [14], the full orthogonality between the classes $\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat})$ and $\mathbf{K}_{ac}(\text{Cotor})$. If $\mathbf{K}(\text{Flat-Cotor})$ is the homotopy category of flat cotorsion modules, then the pair $(\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}), \mathbf{K}_{ac}(\text{Flat-Cotor}))$ is a Bousfield localizing pair in the homotopy category $\mathbf{K}(\text{Flat})$. In other words, the following properties (relative versions of properties (α), (β) and (γ) above) hold:

(α-flat) A complex of flat modules X is K-flat if and only if $\text{Hom}_{\mathbf{K}(R)}(X, Y) = 0$ for all acyclic complexes of flat cotorsion modules Y, whereas a complex of flat modules Y is homotopy equivalent to an acyclic complexes of flat cotorsion modules if and only if $\text{Hom}_{\mathbf{K}(R)}(X, Y) = 0$ for all K-flat complexes of flat modules X.

(β-flat) Both $\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat})$ and $\mathbf{K}_{ac}(\text{Flat-Cotor})$ are thick triangulated subcategories of the homotopy category $\mathbf{K}(\text{Flat})$ of flat modules.

(γ-flat) For any complex of flat modules Z, there is a distinguished triangle in $\mathbf{K}(\text{Flat})$

$$X \longrightarrow Z \longrightarrow Y \longrightarrow SX,$$

where X is a K-flat complex of flat modules and Y an acyclic complex of flat cotorsion modules. It follows that the inclusions $\text{PInj} \subseteq \text{Cotor} \supseteq \text{Flat-Cotor}$ in the category of R-modules induce inclusions $\mathbf{K}_{ac}(\text{PInj}) \subseteq \mathbf{K}_{ac}(\text{Cotor}) \supseteq \mathbf{K}_{ac}(\text{Flat-Cotor})$ in the homotopy category $\mathbf{K}(R)$, which enjoy a weak density property: Complementing property (α-flat) above, it turns out that a complex of flat modules X is K-flat if and only if $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y)=0$ for all acyclic complexes of cotorsion modules Y or, equivalently, for all acyclic complexes of pure injective modules Y.

The contents of the paper are as follows: In Section 1, we give basic definitions and record preliminary results that are used in the sequel. In the following Section, we present a detailed proof of Stovicek's orthogonality criterion. Then, in Sections 3 and 4, we apply this criterion and establish the existence of the two Bousfield localizing pairs mentioned above in the full homotopy category of the ring and the homotopy category of flat modules, respectively.

Notations and terminology. Throughout this paper, R is a fixed unital associative ring. Unless otherwise specified, all modules are left R-modules and all complexes are cochain complexes of left R-modules. If $X=((X^i)_i,\partial)$ is a complex and $n\in\mathbb{Z}$ is any integer, then the n-th translate S^nX of X is the complex whose module of i-cochains is X^{i+n} for all i and whose differential is $(-1)^n\partial$. For any class $\mathfrak{X}$ of modules, we denote by $C(\mathfrak{X})$ the class of those complexes consisting of modules from $\mathfrak{X}$ in each degree. If X,Y and Z are three complexes, then any two cochain maps $f:X\longrightarrow Y$ and $g:X\longrightarrow Z$ induce the cochain map $(f,g):X\longrightarrow Y\oplus Z$, which is given by $x\mapsto(f(x),g(x))$, $x\in X$. Analogously, any two cochain maps $k:X\longrightarrow Z$ and $l:Y\longrightarrow Z$ induce the cochain map $[k,l]:X\oplus Y\longrightarrow Z$, which is given by $(x,y)\mapsto k(x)+l(y)$, $(x,y)\in X\oplus Y$. Let $\mathfrak{C}$ be a category, $\mathfrak{D}\subseteq\mathfrak{C}$ a full subcategory and consider an object $C\in\mathfrak{C}$. Then, a morphism $f:D\longrightarrow C$ is a $\mathfrak{D}$-precover of C if $D\in\mathfrak{D}$ and any morphism $D_0\longrightarrow C$ factors through f for all $D_0\in\mathfrak{D}$. If any object of $\mathfrak{C}$ has a $\mathfrak{D}$-precover, we say that $\mathfrak{D}$ is a precovering subcategory of $\mathfrak{C}$.

1. Preliminaries

In this section, we collect basic notions and record preliminary results that will be used later on. We briefly discuss purity, the homotopy category of the ring and continuous ascending filtrations on complexes.

I. Purity. The notion of purity, which is due to Cohn [9], is closely related to flatness and has been successfully used in several homological algebra problems, including the solution of the flat cover conjecture [6], The reader is advised to consult [29] and the monograph [17] for a detailed account on this notion. A short exact sequence of modules

$$0\longrightarrow M'\xrightarrow{\iota}M\xrightarrow{p}M''\longrightarrow 0$$

is called pure if it remains exact upon tensoring with any right module N; in that case, we also say that ι (resp. p) is a pure monomorphism (resp. a pure epimorphism). Equivalently, the short exact sequence above is pure if it remains exact after applying the functor $\mathrm{Hom}_R(L,\_)$ for any finitely presented module L. An acyclic complex X with cosyzygy modules $(Z^n)_n$ is called pure acyclic if the associated short exact sequences of modules

$$0\longrightarrow Z^n\longrightarrow X^n\longrightarrow Z^{n+1}\longrightarrow 0$$

are pure exact for all n.

A module P is called pure projective if the functor $\mathrm{Hom}_R(P,\_)$ preserves the exactness of any pure exact sequence. In other words, P is pure projective if any pure epimorphism p as above induces an epimorphism of abelian groups $p_*:\mathrm{Hom}_R(P,M)\longrightarrow\mathrm{Hom}_R(P,M'')$. All finitely presented modules are pure projective. In fact, any pure projective module is a direct summand of a suitable direct sum of finitely presented modules. Dually, a module I

is called pure injective if the functor $\mathrm{Hom}_R(\_, I)$ preserves the exactness of any pure exact sequence. In other words, I is pure injective if any pure monomorphism ι as above induces an epimorphism of abelian groups $\iota^* : \mathrm{Hom}_R(M, I) \longrightarrow \mathrm{Hom}_R(M', I)$. We may obtain examples of pure injective modules, by using the Pontryagin duality functor D from the category of left (resp. right) modules to the category of right (resp. left) modules, which is defined by letting $DM = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. It turns out that for any right module M the (left) module DM is pure injective. In fact, any pure injective module is a direct summand of DM for a suitable right module M.

II. Complexes. The homotopy category $\mathbf{K}(R)$ of R has objects all complexes of modules and morphisms the homotopy equivalence classes of cochain maps between them. For any class of modules $\mathfrak{X}$, we denote by $\mathbf{K}(\mathfrak{X})$ the full subcategory of $\mathbf{K}(R)$ consisting of all complexes which are homotopy equivalent to a complex in $C(\mathfrak{X})$. The particular examples of classes $\mathfrak{X}$ that we are interested in are: the class Flat of flat modules, the class Cotor of cotorsion modules, the class Flat-Cotor of flat cotorsion modules, the class PProj of pure projective modules and the class PInj of pure injective modules.

If X, Y are two complexes, then the triviality of the group $\mathrm{Hom}_{\mathbf{K}(R)}(X, Y)$ is equivalent to the assertion that any cochain map $X \longrightarrow Y$ is null-homotopic. In that case, we say that X is left orthogonal to Y and Y is right orthogonal to X in the homotopy category $\mathbf{K}(R)$. Let $\mathfrak{A}$ be a class of cochain complexes. Then, the left Hom-orthogonal of $\mathfrak{A}$ in $\mathbf{K}(R)$ is the class ${}^{\perp}\mathfrak{A}$ consisting of all complexes X, which are left orthogonal to all complexes in $\mathfrak{A}$. The right Hom-orthogonal of $\mathfrak{A}$ in $\mathbf{K}(R)$ is the class $\mathfrak{A}^{\perp}$ consisting of all complexes Y, which are right orthogonal to all complexes in $\mathfrak{A}$. We say that a pair of classes $(\mathfrak{A}, \mathfrak{B})$ of cochain complexes forms an orthogonal pair in $\mathbf{K}(R)$ if $\mathfrak{A} = {}^{\perp}\mathfrak{B}$ and $\mathfrak{A}^{\perp} = \mathfrak{B}$. A triangulated subcategory $\mathfrak{A} \subseteq \mathbf{K}(R)$ is called thick if it has the following property: whenever X, Y are two complexes such that $X \oplus Y \in \mathfrak{A}$, then $X, Y \in \mathfrak{A}$. Any triangulated subcategory with countable products or coproducts is thick. A pair of thick subcategories $(\mathfrak{A}, \mathfrak{B})$ of $\mathbf{K}(R)$ is a Bousfield localizing pair in $\mathbf{K}(R)$ if $\mathfrak{A} \subseteq {}^{\perp}\mathfrak{B}$ (or, equivalently, $\mathfrak{B} \subseteq \mathfrak{A}^{\perp}$) and for any complex Z there exists a distinguished triangle in $\mathbf{K}(R)$

$$A \longrightarrow Z \longrightarrow B \longrightarrow SA,$$

where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. In that case, $(\mathfrak{A}, \mathfrak{B})$ is actually an orthogonal pair in $\mathbf{K}(R)$ and the inclusion functor $\mathfrak{A} \hookrightarrow \mathbf{K}(R)$ (resp. $\mathfrak{B} \hookrightarrow \mathbf{K}(R)$) admits a right (resp. left) adjoint; cf. [23, Chapter 9], [18, §4.9]. As an example, we note that Stovicek proved in [26, Corollary 5.8] that

$$(\mathbf{K}(\mathrm{PProj}, \mathbf{K}_{pac}(R)) \quad \text{and} \quad (\mathbf{K}_{pac}(R), \mathbf{K}(\mathrm{PInj}))$$

are two Bousfield localizing pairs in the homotopy category $\mathbf{K}(R)$. Here, $\mathbf{K}_{pac}(R)$ denotes the triangulated subcategory of $\mathbf{K}(R)$ consisting of the pure acyclic complexes.

If $X = ((X^n)_n, \partial_X)$ and $Y = ((Y^n)_n, \partial_Y)$ are two complexes, the Hom-complex $\mathrm{Hom}_R(X, Y)$ is a complex of abelian groups, which is given in degree n by the group $\prod_i \mathrm{Hom}_R(X^i, Y^{i+n})$ of all homogeneous maps $X \longrightarrow Y$ of degree n. The differential of any n-cochain f is the graded commutator $[\partial, f] = \partial_Y f - (-1)^n f \partial_X$. The 0-cocycles are precisely the cochain maps $X \longrightarrow Y$, whereas the 0-coboundaries are those cochain maps which are null-homotopic; it follows that $H^0\mathrm{Hom}_R(X, Y) = \mathrm{Hom}_{\mathbf{K}(R)}(X, Y)$. For any $n \in \mathbb{Z}$ the cohomology $H^n\mathrm{Hom}_R(X, Y)$ is equal to the group $\mathrm{Hom}_{\mathbf{K}(R)}(X, S^nY)$, where S^nY is the n-th translate of Y.

Spaltenstein defined in [25] a complex P to be K-projective if the Hom-complex $\mathrm{Hom}_R(P, C)$ is acyclic for any acyclic complex C, i.e. if P is contained in the left orthogonal ${}^{\perp}\mathbf{K}_{ac}(R)$ of the class $\mathbf{K}_{ac}(R)$ of acyclic complexes. Any complex X admits a quasi-isomorphism $P \longrightarrow X$ from a K-projective complex of projective modules P. Dually, one defines a complex I to be K-injective if the Hom-complex $\mathrm{Hom}_R(C, I)$ is acyclic for any acyclic complex C, i.e. if I is contained in the right orthogonal $\mathbf{K}_{ac}(R)^{\perp}$ of the class of acyclic complexes. Any complex X admits a quasi-isomorphism $X \longrightarrow I$ to a K-injective complex of injective modules I. As a consequence, the pairs

$$(\mathbf{K}\text{-proj}, \mathbf{K}_{ac}(R)) \quad \text{and} \quad (\mathbf{K}_{ac}(R), \mathbf{K}\text{-inj})$$

are Bousfield localizing pairs in $\mathbf{K}(R)$. Here, we denote by $\mathbf{K}$-proj and $\mathbf{K}$-inj the triangulated subcategories of $\mathbf{K}(R)$ that consist of the K-projective and K-injective complexes respectively.

A complex F is called K-flat if the complex of abelian groups $L \otimes_R F$ is acyclic for any acyclic complex of right modules L. All K-projective complexes are K-flat. Let L be an acyclic complex of right modules. The Hom-tensor duality isomorphism of complexes $D(L \otimes_R F) \simeq \mathrm{Hom}_R(L, DF)$ shows that $L \otimes_R F$ is acyclic if and only if $\mathrm{Hom}_R(L, DF)$ is acyclic. It follows that F is K-flat if and only if the Pontryagin dual complex of right modules DF is K-injective. The acyclic K-flat complexes are precisely the pure acyclic complexes [11, Proposition 1.1], whereas the K-flat complexes of flat modules are the semi-flat complexes studied in [8]. We denote by $\mathbf{K}$-flat the triangulated subcategory of $\mathbf{K}(R)$ consisting of the K-flat complexes.

III. ASCENDING FILTRATIONS. Let X be a complex. An ascending chain of subcomplexes of X is a family of subcomplexes $(X_\alpha)_{\alpha<\gamma}$ of X, which is indexed by an ordinal number γ, such that $X_\alpha \subseteq X_\beta$ whenever α, β are two ordinals with $\alpha < \beta < \gamma$. We say that the filtration is continuous if $X_\alpha = \bigcup_{\alpha' < \alpha} X_{\alpha'}$ whenever $\alpha < \gamma$ is a limit ordinal.

The proof of the following result uses a variation of the argument that is usually employed in order to prove Eklof's lemma.

Proposition 1.1. *Let X, Y be two complexes and consider an ordinal number γ. We assume that X is endowed with a continuous ascending chain of subcomplexes $(X_\alpha)_{\alpha<\gamma}$, such that the following conditions are satisfied:*
(i) $X_0 = 0$ and $X = \bigcup_{\alpha<\gamma} X_\alpha$,
(ii) $Hom_{\mathbf{K}(R)}(X_{\alpha+1}/X_\alpha, S^n Y) = 0$ for all ordinals α with $\alpha + 1 < \gamma$ and all $n \in \mathbb{Z}$ and
(iii) the embedding $X_\alpha^m \hookrightarrow X_{\alpha+1}^m$ of submodules of X^m induces an epimorphism of abelian groups $Hom_R(X_{\alpha+1}^m, Y^n) \longrightarrow Hom_R(X_\alpha^m, Y^n)$ for all α with $\alpha + 1 < \gamma$ and all $n, m \in \mathbb{Z}$.
Then, $Hom_{\mathbf{K}(R)}(X, S^n Y) = 0$ for all $n \in \mathbb{Z}$.

Proof. We have to prove that the Hom-complex $\mathrm{Hom}_R(X, Y)$ is acyclic, i.e. that its n-th cohomology group is trivial for all $n \in \mathbb{Z}$. Since the hypotheses and the statement to be proved are both invariant under replacing Y by any of its translates, it only suffices to consider the case where $n = 0$. In other words, we have to prove that any cochain map $f : X \longrightarrow Y$ is null-homotopic. We fix such a cochain map f and let $f_\alpha = f|_{X_\alpha} : X_\alpha \longrightarrow Y$ be its restriction to the subcomplex $X_\alpha \subseteq X$ for all $\alpha < \gamma$. We shall construct by induction a family of linear maps $\Sigma_\alpha : X_\alpha \longrightarrow Y$, $\alpha < \gamma$, such that the following conditions are satisfied:
(a) The linear map $\Sigma_\alpha : X_\alpha \longrightarrow Y$ is a homotopy from f_α to the zero map, i.e. it is homogeneous of degree -1 and $\partial_Y \Sigma_\alpha + \Sigma_\alpha \partial_\alpha = f_\alpha : X_\alpha \longrightarrow Y$ for any $\alpha < \gamma$. (Here, we denote by ∂_α the differential of X_α.)
(b) For any two ordinals α, β with $\alpha < \beta < \gamma$, we have $\Sigma_\beta|_{X_\alpha} = \Sigma_\alpha : X_\alpha \longrightarrow Y$.
Since $X_0 = 0$, we can only let $\Sigma_0 = 0$. We now assume that $\alpha < \gamma$ is an ordinal and we have

already constructed the partial homotopies $\Sigma_{\alpha'}$ for all ordinals $\alpha' < \alpha$, so that properties (a) and (b) above hold for these.

Assume that α is a limit ordinal, so that $X_\alpha = \bigcup_{\alpha' < \alpha} X_{\alpha'}$. Then, property (b) for the partial homotopies that are already defined shows that there exists a linear map $\Sigma_\alpha : X_\alpha \longrightarrow Y$ of degree -1, which extends $\Sigma_{\alpha'}$ for all $\alpha' < \alpha$. Since the differential ∂_α of X_α restricts to the differential $\partial_{\alpha'}$ of $X_{\alpha'}$ for all $\alpha' < \alpha$, property (a) for the partial homotopies $\Sigma_{\alpha'}$, $\alpha' < \alpha$, implies that $\partial_Y \Sigma_\alpha + \Sigma_\alpha \partial_\alpha = f_\alpha : X_\alpha \longrightarrow Y$, i.e. Σ_α is a homotopy from f_α to the zero map.

We now assume that $\alpha = \beta + 1$ is a successor ordinal and consider the short exact sequence of complexes

$$0 \longrightarrow X_\beta \xrightarrow{\iota} X_{\beta+1} \xrightarrow{\pi} X_{\beta+1}/X_\beta \longrightarrow 0,$$

where ι (resp. π) denotes the inclusion (resp. the quotient map). Then, property (iii) implies that there is an induced short exact sequence of Hom-complexes

$$0 \longrightarrow \mathrm{Hom}_R(X_{\beta+1}/X_\beta, Y) \xrightarrow{\pi^*} \mathrm{Hom}_R(X_{\beta+1}, Y) \xrightarrow{\iota^*} \mathrm{Hom}_R(X_\beta, Y) \longrightarrow 0.$$

The homotopy Σ_β from the restriction f_β to the zero map is a certain homogeneous linear map $X_\beta \longrightarrow Y$ of degree -1. Since the map ι^* is onto, it follows that there exists a homogeneous linear map $\Phi : X_{\beta+1} \longrightarrow Y$ of degree -1, such that $\Phi\iota = \Sigma_\beta$ (i.e. such that $\Phi|_{X_\beta} = \Sigma_\beta$). Then, the cochain map $g = f_{\beta+1} - (\partial_Y \Phi + \Phi \partial_{\beta+1}) : X_{\beta+1} \longrightarrow Y$ vanishes on X_β, since $f_\beta = \partial_Y \Sigma_\beta + \Sigma_\beta \partial_\beta$, and hence defines by passage to the quotient a cochain map $h : X_{\beta+1}/X_\beta \longrightarrow Y$, so that $g = h\pi$. In view of property (ii), we know that the cochain map h is null-homotopic; thus, there exists a homogeneous linear map $T : X_{\beta+1}/X_\beta \longrightarrow Y$ of degree -1, such that $h = \partial_Y T + T\overline{\partial}$ (where $\overline{\partial}$ denotes the differential of the quotient complex $X_{\beta+1}/X_\beta$). We now consider the homogeneous linear map $\Sigma_{\beta+1} = \Phi + T\pi : X_{\beta+1} \longrightarrow Y$ of degree -1 and compute

$$\begin{aligned}
\partial_Y \Sigma_{\beta+1} + \Sigma_{\beta+1}\partial_{\beta+1} &= \partial_Y(\Phi + T\pi) + (\Phi + T\pi)\partial_{\beta+1} \\
&= \partial_Y \Phi + \Phi\partial_{\beta+1} + \partial_Y T\pi + T\pi\partial_{\beta+1} \\
&= \partial_Y \Phi + \Phi\partial_{\beta+1} + \partial_Y T\pi + T\overline{\partial}\pi \\
&= \partial_Y \Phi + \Phi\partial_{\beta+1} + (\partial_Y T + T\overline{\partial})\pi \\
&= \partial_Y \Phi + \Phi\partial_{\beta+1} + h\pi \\
&= \partial_Y \Phi + \Phi\partial_{\beta+1} + g \\
&= f_{\beta+1}.
\end{aligned}$$

Therefore, $\Sigma_{\beta+1}$ is a homotopy from $f_{\beta+1}$ to the zero map. Since the restriction of $\Sigma_{\beta+1}$ to the subcomplex $X_\beta \subseteq X_{\beta+1}$ is the map $\Sigma_{\beta+1}\iota = (\Phi + T\pi)\iota = \Phi\iota + T\pi\iota = \Phi\iota + T0 = \Phi\iota = \Sigma_\beta$, the inductive step of the construction is complete.

Since $X = \bigcup_{\alpha < \gamma} X_\alpha$, property (b) for the partial homotopies Σ_α that we have constructed shows that there exists a linear map $\Sigma : X \longrightarrow Y$ of degree -1, which extends Σ_α for all $\alpha < \gamma$. Since the differential ∂_X of X restricts to the differential ∂_α of X_α for all $\alpha < \gamma$, property (a) for the partial homotopies Σ_α, $\alpha < \gamma$, implies that we also have $\partial_Y \Sigma + \Sigma \partial_X = f : X \longrightarrow Y$. We have therefore constructed a homotopy from f to the zero map, as needed. □

2. Stovicek's orthogonality criterion

In this section, we prove in detail a result which is essentially due to Stovicek [26], concerning the orthogonality in the homotopy category $\mathbf{K}(R)$. The particular form of Stovicek's criterion that is presented here is strongly reminiscent of a result by Bazzoni et al. [5].

We consider a filtered direct system of complexes $(X_i)_{i\in I}$ with structure maps $f_{ij} : X_i \longrightarrow X_j$ for all $i \leq j$ and the colimit $X = \mathrm{colim}_i X_i$. We also consider a complex $Y = ((Y^n)_n, \partial_Y)$ and a class $\mathfrak{X}$ of modules, such that the following conditions are satisfied:
(a) $\mathrm{Hom}_{\mathbf{K}(R)}(X_i, S^nY) = 0$ for all $i \in I$ and all $n \in \mathbb{Z}$,
(b) $X_i \in C(\mathfrak{X})$ for all $i \in I$, i.e. all of the X_i's are complexes of modules in $\mathfrak{X}$,
(c) $\mathfrak{X}$ is closed under direct sums and filtered colimits and
(d) any pure monomorphism of modules $\iota : M \longrightarrow M'$ with $M, M' \in \mathfrak{X}$ induces an epimorphism of abelian groups $\iota^* : \mathrm{Hom}_R(M', Y^n) \longrightarrow \mathrm{Hom}_R(M, Y^n)$ for all $n \in \mathbb{Z}$.
Then, Stovicek's result can be stated as follows.

Theorem 2.1. *With notation and hypotheses as above, we also have $Hom_{\mathbf{K}(R)}(X, S^nY) = 0$ for all $n \in \mathbb{Z}$.*

We note that there is a short exact sequence of complexes

$$0 \longrightarrow K \longrightarrow \textstyle\bigoplus_i X_i \xrightarrow{p} X \longrightarrow 0, \tag{1}$$

which is pure exact in each degree. Here, p is the cochain map induced by the canonical maps $X_i \longrightarrow X$, $i \in I$. The proof of Theorem 2.1 will occupy the remaining of the section and is based on Stovicek's arguments. For expository purposes, we shall proceed in three steps.

I. Expressing K as a directed union. The subcomplex $K \subseteq \bigoplus_i X_i$ is the image of the cochain map

$$h : \textstyle\bigoplus_{i<j} X_i \longrightarrow \bigoplus_i X_i,$$

whose composition with the natural inclusion $\nu_{ij} : X_i \hookrightarrow \bigoplus_{i<j} X_i$ that corresponds to a pair of indices (i,j) with $i<j$, is the composition of $(1_{X_i}, -f_{ij}) : X_i \longrightarrow X_i \oplus X_j$ and the natural inclusion $X_i \oplus X_j \hookrightarrow \bigoplus_i X_i$. For any pair of indices (i,j) with $i<j$, we denote by $K(i,j)$ the image of the latter composition, i.e. we let

$$K(i,j) = \mathrm{im}\left(X_i \xrightarrow{\nu_{ij}} \textstyle\bigoplus_{i<j} X_i \xrightarrow{h} \bigoplus_i X_i \right).$$

Then, $K = \sum_{i<j} K(i,j)$. We note that if $i,j,k \in I$ are three indices with $i<j<k$, then $K(i,j) \subseteq K(i,k) + K(j,k)$; this is an immediate consequence of the equality $f_{ik} = f_{jk} f_{ij}$.

We say that a finite subset $c \subseteq I$ is a cone if c has a maximum element. If c is a cone with maximum element j and $c \setminus \{j\} = \{i_1, \ldots, i_n\}$, then we write $K(c) = \sum_{t=1}^n K(i_t, j) \subseteq K$.

Lemma 2.2. *Let C be the set of all cones as above. Then:*
(i) The set C is directed with respect to the ordinary inclusion of sets.
(ii) For any pair of indices (i,j) with $i<j$ there exists a cone $c \in C$, such that $K(i,j) \subseteq K(c)$.
(iii) If $c, c' \in C$ are two cones with $c \subseteq c'$, then $K(c) \subseteq K(c')$.
(iv) $K(c)$ is a complex of modules in $\mathfrak{X}$, i.e. $K(c) \in C(\mathfrak{X})$ for any cone $c \in C$.
(v) For any cone $c \in C$ the embedding $K(c) \hookrightarrow K$ is a split monomorphism of complexes.
(vi) For any cone $c \in C$ we have $Hom_{\mathbf{K}(R)}(K(c), S^nY) = 0$ for all $n \in \mathbb{Z}$.

Proof. (i) This is clear since any finite subset $F \subseteq I$ has an upper bound $j \in I$ and $F \cup \{j\}$ is then a cone.

(ii) Given a pair of indices (i,j) with $i<j$, we may let $c = \{i,j\}$.

(iii) Let c, c' be two cones with $c \subseteq c'$ and consider their maximum elements j, j' respectively. Then, we have to show that $K(i,j) \subseteq K(c')$ for all $i \in c \setminus \{j\}$. This is obvious if $j = j'$ and hence we may assume that $j < j'$. But then, for any $i \in c \setminus \{j\}$ we have $i < j < j'$ and hence $K(i,j) \subseteq K(i,j') + K(j,j') \subseteq K(c')$, as needed.

(iv), (v), (vi) Let c be a cone with maximum element j and $c \setminus \{j\} = \{i_1, \ldots, i_n\}$. We note that the composition

$$X_{i_1} \oplus \ldots \oplus X_{i_n} \xrightarrow{\nu} \bigoplus_{i<j} X_i \xrightarrow{h} \bigoplus_i X_i \hookrightarrow \prod_i X_i \xrightarrow{\pi} \prod_{t=1}^n X_{i_t} = X_{i_1} \oplus \ldots \oplus X_{i_n},$$

where $\nu = [\nu_{i_1 j}, \ldots, \nu_{i_n j}]$ and π is the canonical projection, is the identity. It follows that $K(c) = \operatorname{im} h\nu \simeq \bigoplus_{t=1}^n X_{i_t}$ and the embedding $K(c) \hookrightarrow \bigoplus_i X_i$ splits. Since $X_{i_t} \in C(\mathfrak{X})$ for all $t = 1, \ldots, n$ and $\mathfrak{X}$ is closed under direct sums, it follows that $K(c) \in C(\mathfrak{X})$ as well; this proves (iv). Moreover, since $K(c) \subseteq K \subseteq \bigoplus_i X_i$, the embedding $K(c) \hookrightarrow K$ is also split and this proves (v). On the other hand, the Hom-complex $\mathrm{Hom}_R(K(c), Y) \simeq \bigoplus_{t=1}^n \mathrm{Hom}_R(X_{i_t}, Y)$ is acyclic, since all of the $\mathrm{Hom}_R(X_{i_t}, Y)$'s are. It follows that $\mathrm{Hom}_{\mathbf{K}(R)}(X(c), S^n Y) = 0$ for all $n \in \mathbb{Z}$ and hence (vi) is also proved. □

Since $K = \sum_{i<j} K(i,j)$, Lemma 2.2 implies that K is the union of the directed family of subcomplexes $(K(c))_{c \in C}$. Moreover, for any $c \in C$ the complex $K(c)$ has the following properties:
I-(i) $K(c) \in C(\mathfrak{X})$,
I-(ii) the embedding $K(c) \hookrightarrow K$ is a pure monomorphism in each degree and
I-(iii) $\mathrm{Hom}_{\mathbf{K}(R)}(K(c), S^n Y) = 0$ for all $n \in \mathbb{Z}$.

II. Expressing K as the union of an ascending chain. We shall prove that any complex K, which can be expressed as the union of a directed family of subcomplexes $(K(c))_{c \in C}$, in such a way that properties I-(i), I-(ii) and I-(iii) above hold, is itself contained in the left orthogonal of the set $\{S^n Y : n \in \mathbb{Z}\}$ in $\mathbf{K}(R)$. To that end, we use induction on the cardinality of the directed set C.

There is nothing to prove if the set C is finite; in that case, C has a maximum element c' and $K = \operatorname{colim}_c K(c) = K(c')$. The result is also clear if C is countable. If $C = \{c_0, c_1, c_2, \ldots\}$, then there is a short exact sequence of complexes

$$0 \longrightarrow \bigoplus_{t=0}^{\infty} K(c_t) \longrightarrow \bigoplus_{t=0}^{\infty} K(c_t) \xrightarrow{p} K \longrightarrow 0,$$

which is pure exact in each degree. Here, p is the cochain map induced by the inclusion maps $K(c_t) \hookrightarrow K$, $t \geq 0$. Since $K(c_t) \in C(\mathfrak{X})$ and $\mathfrak{X}$ is closed under direct sums, it follows that $\bigoplus_{t=0}^{\infty} K(c_t) \in C(\mathfrak{X})$. Then, assumption (d) at the beginning of §2 implies that there is an induced short exact sequence of Hom-complexes

$$0 \longrightarrow \mathrm{Hom}_R(K, Y) \xrightarrow{p^*} \mathrm{Hom}_R(\bigoplus_{t=0}^{\infty} K(c_t), Y) \longrightarrow \mathrm{Hom}_R(\bigoplus_{t=0}^{\infty} K(c_t), Y) \longrightarrow 0.$$

Since the complexes $\mathrm{Hom}_R(K(c_t), Y)$ are acyclic for all $t \geq 0$, it follows that the complex $\mathrm{Hom}_R(\bigoplus_{t=0}^{\infty} K(c_t), Y) = \prod_{t=0}^{\infty} \mathrm{Hom}_R(K(c_t), Y)$ is also acyclic. Therefore, $\mathrm{Hom}_R(K, Y)$ must be an acyclic complex as well and hence $\mathrm{Hom}_{\mathbf{K}(R)}(K, S^n Y) = 0$ for all $n \in \mathbb{Z}$, as needed.

We now assume that C is uncountable of cardinality γ and that the result is true for unions of directed families of subcomplexes over directed sets of cardinality $< \gamma$. We choose a function $\lambda : C \times C \longrightarrow C$, such that $\lambda(c, c') \geq c, c'$ for all $c, c' \in C$. Following Adamek and Rosicky [1, §1.6], for any subset $D \subseteq C$ we let $\overline{D} = D \cup \{\lambda(c, c') : c, c' \in D\} \subseteq C$. We note that $\overline{D}$ is finite if D is finite; if D is infinite, then $\operatorname{card} \overline{D} = \operatorname{card} D$. Given $D \subseteq C$, we define the increasing sequence $(D_n)_n$ of subsets of C, by letting $D_0 = D$ and $D_n = \overline{D_{n-1}}$ for all $n \geq 1$. Then, the union $D^+ = \bigcup_n D_n$ is a directed subset of C with $D^+ \supseteq D$ and $\operatorname{card} D^+ \leq \max\{\operatorname{card} D, \aleph_0\}$.

We may index the elements of C by $\gamma = \{\alpha : \alpha < \gamma\}$ and write $C = \{c_\alpha : \alpha < \gamma\}$. We now construct a continuous ascending chain of directed subsets $(C_\alpha)_{\alpha < \gamma}$ of C, by letting $C_0 = \emptyset$,

$C_{\alpha+1} = (C_\alpha \cup \{c_\alpha\})^+$ if $\alpha + 1 < \gamma$ is a successor ordinal and $C_\alpha = \bigcup_{\alpha'<\alpha} C_{\alpha'}$ if $\alpha < \gamma$ is a limit ordinal. Since $c_\alpha \in C_{\alpha+1}$ for any $\alpha < \gamma$, it follows that $C = \bigcup_{\alpha<\gamma} C_\alpha$. Using induction on α, we can prove that $\operatorname{card} C_\alpha \leq \max\{\operatorname{card}\alpha, \aleph_0\}$ for any $\alpha < \gamma$; in particular, we conclude that $\operatorname{card} C_\alpha < \gamma$ for any $\alpha < \gamma$. We can use this continuous ascending chain of directed subsets of C, in order to define an ascending chain of subcomplexes $(K_\alpha)_{\alpha<\gamma}$ of K, by letting $K_\alpha = \operatorname{colim}_{c\in C_\alpha} K(c) = \bigcup_{c\in C_\alpha} K(c)$.

Lemma 2.3. *Let the notation be as above. Then:*
(i) $(K_\alpha)_{\alpha<\gamma}$ is a continuous ascending chain of subcomplexes of K,
(ii) $K_0 = 0$ and $K = \operatorname{colim}_{\alpha<\gamma} K_\alpha = \bigcup_{\alpha<\gamma} K_\alpha$,
(iii) $K_\alpha \in C(\mathfrak{X})$ and the embedding $K_\alpha \hookrightarrow K$ is a pure monomorphism in each degree for all $\alpha < \gamma$,
(iv) $\operatorname{Hom}_{\mathbf{K}(R)}(K_\alpha, S^nY) = 0$ for all $n \in \mathbb{Z}$ and $\alpha < \gamma$ and
(v) $\operatorname{Hom}_{\mathbf{K}(R)}(K_{\alpha+1}/K_\alpha, S^nY) = 0$ for all $n \in \mathbb{Z}$ and $\alpha < \gamma$.

Proof. (i) Let $\alpha < \gamma$ be a limit ordinal, so that $C_\alpha = \bigcup_{\alpha'<\alpha} C_{\alpha'}$. Then, $K_\alpha = \bigcup_{c\in C_\alpha} K(c) = \bigcup_{\alpha'<\alpha} \bigcup_{c\in C_{\alpha'}} K(c) = \bigcup_{\alpha'<\alpha} K_{\alpha'}$.

(ii) Since $C_0 = \emptyset$, it is clear that $K_0 = 0$. Since $C = \bigcup_{\alpha<\gamma} C_\alpha$, we have $K = \bigcup_{c\in C} K(c) = \bigcup_{\alpha<\gamma} \bigcup_{c\in C_\alpha} K(c) = \bigcup_{\alpha<\gamma} K_\alpha$.

(iii) Let $\alpha < \gamma$ be an ordinal. Then, $K_\alpha = \operatorname{colim}_{c\in C_\alpha} K(c)$ is the filtered colimit of complexes contained in $C(\mathfrak{X})$. Since $\mathfrak{X}$ is closed under filtered colimits, it follows that $K_\alpha \in C(\mathfrak{X})$ as well. Moreover, since the embedding $K(c) \hookrightarrow K$ is a pure monomorphism in each degree for all $c \in C$, it follows that the embedding $K_\alpha \hookrightarrow K$ is also a pure monomorphism in each degree.

(iv) We recall that for any ordinal $\alpha < \gamma$ we have $\operatorname{card} C_\alpha < \gamma$. Since $K_\alpha = \bigcup_{c\in C_\alpha} K(c)$ is expressed as the union of a directed family of subcomplexes that satisfy properties I-(i), I-(ii) and I-(iii), we may use the induction hypothesis and conclude that $\operatorname{Hom}_{\mathbf{K}(R)}(K_\alpha, S^nY) = 0$ for all $n \in \mathbb{Z}$.

(v) Let $\alpha < \gamma$ be an ordinal. Since the embedding $K_\alpha \hookrightarrow K$ is a pure monomorphism in each degree, the short exact sequence of complexes

$$0 \longrightarrow K_\alpha \longrightarrow K_{\alpha+1} \longrightarrow K_{\alpha+1}/K_\alpha \longrightarrow 0$$

is pure exact in each degree. Since $K_\alpha, K_{\alpha+1} \in C(\mathfrak{X})$, assumption (d) at the beginning of §2 implies that there is an induced short exact sequence of Hom-complexes

$$0 \longrightarrow \operatorname{Hom}_R(K_{\alpha+1}/K_\alpha, Y) \longrightarrow \operatorname{Hom}_R(K_{\alpha+1}, Y) \longrightarrow \operatorname{Hom}_R(K_\alpha, Y) \longrightarrow 0.$$

In view of (iv), the complexes $\operatorname{Hom}_R(K_\alpha, Y)$ and $\operatorname{Hom}_R(K_{\alpha+1}, Y)$ are both acyclic. It follows that the complex $\operatorname{Hom}_R(K_{\alpha+1}/K_\alpha, Y)$ is acyclic as well, i.e. $\operatorname{Hom}_{\mathbf{K}(R)}(K_{\alpha+1}/K_\alpha, S^nY) = 0$ for all $n \in \mathbb{Z}$. $\square$

Having proved Lemma 2.3, we note that K is expressed as the union of a continuous ascending chain of subcomplexes $(K_\alpha)_{\alpha<\gamma}$, such that $K_0 = 0$ and for any ordinal $\alpha < \gamma$ the following conditions hold:
II-(i) $K_\alpha \in C(\mathfrak{X})$,
II-(ii) the embedding $K_\alpha \hookrightarrow K$ is a pure monomorphism in each degree and
II-(iii) $\operatorname{Hom}_{\mathbf{K}(R)}(K_{\alpha+1}/K_\alpha, S^nY) = 0$ for all $n \in \mathbb{Z}$.

III. Concluding the proof. We may now complete the proof of the induction that started at the beginning of §2.II above and use Proposition 1.1, in order to show that the complex K

is indeed contained in the left orthogonal of the set $\{S^nY : n \in \mathbb{Z}\}$ in the homotopy category $\mathbf{K}(R)$. Since K is the union of a continuous ascending chain of subcomplexes $(K_\alpha)_{\alpha<\gamma}$ with $K_0 = 0$, it suffices to show that properties (ii) and (iii) in the statement of Proposition 1.1 are satisfied. First of all, we note that property (ii) therein is exactly property II-(iii) above. If α is an ordinal with $\alpha < \gamma$, then property II-(ii) above implies that the embedding $K_\alpha \hookrightarrow K_{\alpha+1}$ is a pure monomorphism in each degree. Therefore, property (iii) in the statement of Proposition 1.1 is an immediate consequence of property II-(i) above, in view of assumption (d) that was made on the pair $(\mathfrak{X}, Y)$ at the beginning of §2.

Having proved that K is contained in the left orthogonal of the set $\{S^nY : n \in \mathbb{Z}\}$, we can finish the proof of Theorem 2.1. Since $\mathfrak{X}$ is closed under filtered colimits, Lemma 2.3(ii),(iii) implies that $K \in C(\mathfrak{X})$. Since $\mathfrak{X}$ is also closed under direct sums and $X_i \in C(\mathfrak{X})$, it follows that $\bigoplus_i X_i \in C(\mathfrak{X})$. In view of our assumption (d) on the pair $(\mathfrak{X}, Y)$, the short exact sequence (1), which is pure in each degree, induces a short exact sequence of Hom-complexes

$$0 \longrightarrow \mathrm{Hom}_R(X,Y) \xrightarrow{p^*} \mathrm{Hom}_R(\textstyle\bigoplus_i X_i, Y) \longrightarrow \mathrm{Hom}_R(K,Y) \longrightarrow 0.$$

Since the complexes $\mathrm{Hom}_R(X_i, Y)$ are acyclic for all i, it follows that $\mathrm{Hom}_R(\bigoplus_i X_i, Y) = \prod_i \mathrm{Hom}_R(X_i, Y)$ is also acyclic. As we have proved above that the complex $\mathrm{Hom}_R(K,Y)$ is acyclic, we conclude that the complex $\mathrm{Hom}_R(X,Y)$ must be acyclic as well. In other words, we have proved that $\mathrm{Hom}_{\mathbf{K}(R)}(X, S^nY) = 0$ for all $n \in \mathbb{Z}$, as needed.

3. A description of K-flat complexes via Hom-orthogonality

In this section, we shall apply Theorem 2.1, in order to show that K-flat complexes are left orthogonal to acyclic complexes of pure injective modules. In fact, we show that the classes $\mathbf{K}$-flat and $\mathbf{K}_{ac}(\mathrm{PInj})$ constitute an orthogonal pair in the homotopy category $\mathbf{K}(R)$ and that a Bousfield localization exists for the embedding $\mathbf{K}\text{-flat} \hookrightarrow \mathbf{K}(R)$.

Proposition 3.1. *If X is a K-flat complex and Y an acyclic complex of pure injective modules, then any cochain map $X \longrightarrow Y$ is null-homotopic, i.e. $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y) = 0$.*

Proof. In view of [12, Theorem 3.8], the K-flat complex X is homotopy equivalent to a filtered colimit of K-projective complexes (of pure projective modules). In other words, there exists a filtered direct system of complexes $(X_i)_i$, such that X_i is K-projective for all i and $X \simeq \mathrm{colim}_i X_i$ in the homotopy category $\mathbf{K}(R)$. Since Y is acyclic (and the same is true for all of its translates), we have $\mathrm{Hom}_{\mathbf{K}(R)}(X_i, S^nY) = 0$ for all i and all $n \in \mathbb{Z}$. We can now apply Theorem 2.1, by letting $\mathfrak{X}$ be the class of all modules therein. Then, all of the hypotheses made at the beginning of §2 are satisfied; (b) and (c) are vacuous and (d) holds since Y is a complex of pure injective modules. Therefore, we may conclude that $\mathrm{Hom}_{\mathbf{K}(R)}(X, S^nY) = 0$ for all $n \in \mathbb{Z}$; in particular, $\mathrm{Hom}_{\mathbf{K}(R)}(X,Y) = 0$, as needed. □

Even though the following result is a formal consequence of the existence of approximation triangles that will be established in Theorem 3.4 below, it is perhaps instructive to include a direct proof at this point.

Proposition 3.2. *The pair $(\mathbf{K}\text{-flat}, \mathbf{K}_{ac}(\mathrm{PInj}))$ is an orthogonal pair in $\mathbf{K}(R)$.*

Proof. Since we know that the two classes are orthogonal to each other (cf. Proposition 3.1), it only remains to show that ${}^\perp\mathbf{K}_{ac}(\mathrm{PInj}) \subseteq \mathbf{K}\text{-flat}$ and $\mathbf{K}\text{-flat}^\perp \subseteq \mathbf{K}_{ac}(\mathrm{PInj})$.

We begin by proving the inclusion ${}^\perp\mathbf{K}_{ac}(\mathrm{PInj}) \subseteq \mathbf{K}\text{-flat}$. To that end, consider a complex X which is left orthogonal to the class $\mathbf{K}_{ac}(\mathrm{PInj})$ of all acyclic complexes of pure injective modules. In order to show that X is K-flat, let C be an acyclic complex of right modules. Then, the Pontryagin dual DC is an acyclic complex consisting of pure injective (left) modules. In view of our hypothesis on X, the complex $D(C \otimes_R X) \simeq \mathrm{Hom}_R(X, DC)$ is acyclic and hence the complex $C \otimes_R X$ must be acyclic as well (the abelian group $\mathbb{Q}/\mathbb{Z}$ being an injective cogenerator of the category of abelian groups). As this is the case for any acyclic complex of right modules C, we conclude that X is K-flat, as needed.

In order to prove the inclusion $\mathbf{K}\text{-flat}^{\perp} \subseteq \mathbf{K}_{ac}(\mathrm{PInj})$, we consider a complex Y which is right orthogonal to the class $\mathbf{K}$-flat of all K-flat complexes. Since the complex R (consisting of R in degree 0 and zeroes elsewhere) is K-flat, we conclude that the complex $\mathrm{Hom}_R(R, Y) \simeq Y$ is acyclic. As shown by Stovicek in [26, Corollary 5.8], there exists a distinguished triangle in $\mathbf{K}(R)$

$$Z \longrightarrow Y \xrightarrow{f} W \longrightarrow SZ,$$

where W is a complex of pure injective modules and Z is pure acyclic. Since both Z, Y are acyclic, it follows that W is acyclic as well, i.e. $W \in \mathbf{K}_{ac}(\mathrm{PInj})$. As we have already proved in Proposition 3.1 that $\mathbf{K}_{ac}(\mathrm{PInj}) \subseteq \mathbf{K}\text{-flat}^{\perp}$, we conclude that $W \in \mathbf{K}\text{-flat}^{\perp}$. But Y is also contained in $\mathbf{K}\text{-flat}^{\perp}$ (in view of our assumption) and hence the distinguished triangle above shows that $Z \in \mathbf{K}\text{-flat}^{\perp}$. Being pure acyclic, the complex Z is certainly K-flat. It follows that $Z \in \mathbf{K}\text{-flat} \cap \mathbf{K}\text{-flat}^{\perp}$ and hence $Z \simeq 0$ in the homotopy category $\mathbf{K}(R)$, i.e. Z is contractible. Therefore, the morphism f is represented by a cochain map $\phi : Y \longrightarrow W$, whose mapping cone is contractible. It follows that ϕ is a homotopy equivalence between Y and $W \in \mathbf{K}_{ac}(\mathrm{PInj})$ and hence $Y \in \mathbf{K}_{ac}(\mathrm{PInj})$, as needed. □

Remark 3.3. K-flat complexes were defined by Spaltenstein [25], as those complexes X for which the tensor product complex functor $\_ \otimes_R X$ from the category of complexes of right modules to the category of complexes of abelian groups preserves acyclicity. The equality $\mathbf{K}\text{-flat} = {}^{\perp}\mathbf{K}_{ac}(\mathrm{PInj})$, which is proved in Proposition 3.2, shows that K-flatness can be defined solely in terms of the Hom-pairing in the homotopy category $\mathbf{K}(R)$ and the concept of pure injectivity. The embedding $\mathbf{K}\text{-proj} \subseteq \mathbf{K}\text{-flat}$ is induced by the embedding $\mathbf{K}_{ac}(\mathrm{PInj}) \subseteq \mathbf{K}_{ac}(R)$ by taking left orthogonals in the homotopy category $\mathbf{K}(R)$.

Theorem 3.4. *The pair* $(\mathbf{K}\textit{-flat}, \mathbf{K}_{ac}(\mathit{PInj}))$ *is a Bousfield localizing pair in* $\mathbf{K}(R)$.

Proof. Since the direct sum of any family of K-flat complexes is a K-flat complex, whereas the direct product of any family of acyclic complexes of pure injective modules is an acyclic complex of pure injective modules, the triangulated subcategories $\mathbf{K}$-flat and $\mathbf{K}_{ac}(\mathrm{PInj})$ of the homotopy category $\mathbf{K}(R)$ are thick. They are orthogonal to each other (cf. Proposition 3.1) and hence it only remains to show the existence of approximation triangles.

To that end, we fix a chain complex X and consider a quasi-isomorphism $P \longrightarrow X$, where P is a K-projective complex of projective modules. Its mapping cone Y is acyclic and fits into a distinguished triangle the homotopy category $\mathbf{K}(R)$

$$P \longrightarrow X \xrightarrow{f} Y \longrightarrow SP.$$

Using [26, Corollary 5.8], we conclude that there exists another distinguished triangle in $\mathbf{K}(R)$

$$Z \longrightarrow Y \xrightarrow{g} W \longrightarrow SZ,$$

where W is a complex of pure injective modules and Z is pure acyclic. Since both Z, Y are acyclic, it follows that W is acyclic as well, i.e. $W \in \mathbf{K}_{ac}(\mathrm{PInj})$. We complete the composition

$gf : X \longrightarrow W$ to a distinguished triangle

$$X \xrightarrow{gf} W \longrightarrow V \longrightarrow SX \tag{2}$$

and apply the octahedral axiom to the composable pair of morphisms (f, g)

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & SP & \longrightarrow & SX \\
\downarrow 1_X & & \downarrow g & & & & \downarrow 1_X \\
X & \xrightarrow{gf} & W & \longrightarrow & V & \longrightarrow & SX \\
 & & \downarrow & & & & \downarrow Sf \\
 & & SZ & \xrightarrow{1_{SZ}} & SZ & \longrightarrow & SY \\
 & & \downarrow & & & & \\
 & & SY & \longrightarrow & S^2P & &
\end{array}$$

in order to obtain a distinguished triangle in $\mathbf{K}(R)$

$$SP \longrightarrow V \longrightarrow SZ \longrightarrow S^2P.$$

We note that the complexes P and Z are both K-flat; indeed, P is K-projective and Z is pure acyclic. Since $\mathbf{K}$-flat is a triangulated subcategory of the homotopy category, it follows that V is also K-flat. Then, the distinguished triangle obtained by shifting (2)

$$S^{-1}V \longrightarrow X \xrightarrow{gf} W \longrightarrow V$$

is the approximation triangle we are looking for. □

The following result follows formally from general properties of Bousfield localization (cf. [23, Chapter 9], [18, Proposition 4.9.1]).

Corollary 3.5. *(i) The inclusion functor* $\mathbf{K}$*-flat* $\hookrightarrow \mathbf{K}(R)$ *admits a right adjoint and the inclusion functor* $\mathbf{K}_{ac}(PInj) \hookrightarrow \mathbf{K}(R)$ *admits a left adjoint.*

(ii) Let $\iota_\rho : \mathbf{K}(R) \longrightarrow \mathbf{K}$*-flat be a right adjoint functor to the inclusion* $\mathbf{K}$*-flat* $\hookrightarrow \mathbf{K}(R)$*. Then, for any complex* X *the counit of adjunction morphism* $\iota_\rho X \longrightarrow X$ *is a K-flat precover.*

(iii) Let $\mathbf{K}(R)/\mathbf{K}$*-flat be the Verdier quotient of the homotopy category* $\mathbf{K}(R)$ *by the triangulated subcategory* $\mathbf{K}$*-flat. Then, the quotient functor* $p : \mathbf{K}(R) \longrightarrow \mathbf{K}(R)/\mathbf{K}$*-flat admits a right adjoint and the composition* $\mathbf{K}_{ac}(PInj) \hookrightarrow \mathbf{K}(R) \xrightarrow{p} \mathbf{K}(R)/\mathbf{K}$*-flat is an equivalence of categories. In particular, the quotient* $\mathbf{K}(R)/\mathbf{K}$*-flat has small Hom-sets.* □

If we identify the Verdier quotient $\mathbf{K}(R)/\mathbf{K}$-flat with $\mathbf{K}_{ac}$(PInj), by means of the composition $\mathbf{K}_{ac}(\mathrm{PInj}) \hookrightarrow \mathbf{K}(R) \xrightarrow{p} \mathbf{K}(R)/\mathbf{K}$-flat, then the right adjoint to the quotient functor p is identified with the inclusion functor $\mathbf{K}_{ac}(\mathrm{PInj}) \hookrightarrow \mathbf{K}(R)$ and, of course, the left adjoint to the latter inclusion functor is identified with the quotient functor p.

We may reformulate pictorially some of the assertions made above by the following diagrams of triangulated subcategories of the homotopy category $\mathbf{K}(R)$

$$\begin{array}{ccc}
\mathbf{K}\text{-flat} & \longleftarrow & \mathbf{K}\text{-proj} \\
\uparrow & & \\
\mathbf{K}_{pac}(R) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{K}_{ac}(\mathrm{PInj}) & \longrightarrow & \mathbf{K}_{ac}(R) \\
\downarrow & & \\
\mathbf{K}(\mathrm{PInj}) & &
\end{array}$$

Here, all arrows are inclusions and the left (resp. right) hand side diagram is obtained from the right (resp. left) hand side diagram by taking left (resp. right) Hom-orthogonals. It is clear that $\mathbf{K}_{ac}(\mathrm{PInj}) = \mathbf{K}_{ac}(R) \cap \mathbf{K}(\mathrm{PInj})$ is the biggest triangulated subcategory of the homotopy category, which is contained in both $\mathbf{K}_{ac}(R)$ and $\mathbf{K}(\mathrm{PInj})$. We shall conclude this section, by proving the dual assertion for $\mathbf{K}$-flat. In fact, the following description of the category $\mathbf{K}$-flat may be used for an alternative approach to some of the results obtained earlier in this section.

Proposition 3.6. *The category $\mathbf{K}$-flat is the smallest triangulated subcategory of the homotopy category, which contains both $\mathbf{K}_{pac}(R)$ and $\mathbf{K}$-proj.*

Proof. Let $\mathbf{T} \subseteq \mathbf{K}(R)$ be a triangulated subcategory, containing both $\mathbf{K}_{pac}(R)$ and $\mathbf{K}$-proj, and consider a K-flat complex X. As shown by Stovicek in [26, Corollary 5.8], there exists a distinguished triangle in $\mathbf{K}(R)$

$$Y \longrightarrow X \longrightarrow W \longrightarrow SY, \tag{3}$$

where Y is a complex of pure projective modules and W is pure acyclic. Since $\mathbf{K}_{pac}(R) \subseteq \mathbf{T}$, it follows that $W \in \mathbf{T}$. Since pure acyclic complexes are K-flat, we conclude that both complexes X, W are K-flat; hence, Y is K-flat as well. As shown in [12, Corollary 3.4], any K-flat complex of pure projective complexes is necessarily K-projective. Therefore, we conclude that the complex Y is K-projective and hence $Y \in \mathbf{T}$. The distinguished triangle (3) then shows that $X \in \mathbf{T}$, as needed. □

4. K-flat complexes of flat modules

In this section, we apply Theorem 2.1 once more, in order to examine the relation between the K-flat complexes of flat modules and the acyclic complexes of cotorsion modules. It will turn out that the classes $\mathbf{K}\text{-flat} \cap \mathbf{K}(\mathrm{Flat})$ and $\mathbf{K}_{ac}(\mathrm{Cotor})$ are orthogonal to each other in the homotopy category $\mathbf{K}(R)$. This recovers a result by Bazzoni et al. [5], according to which the cosyzygy modules of any acyclic complex of cotorsion modules are themselves cotorsion. In fact, the latter result implies the full orthogonality assertion made above, in view of Gillespie's work [14]. Then, as a relative version of Theorem 3.4, it will follow that a Bousfield localization exists for the embedding $\mathbf{K}\text{-flat} \cap \mathbf{K}(\mathrm{Flat}) \hookrightarrow \mathbf{K}(\mathrm{Flat})$.

We wish to proceed in a way that parallels the method of §3. In the same way that any K-flat complex can be expressed (up to homotopy equivalence) as a filtered colimit of K-projective complexes of pure projective modules, one can also express (up to homotopy equivalence) any K-flat complex of flat modules as a filtered colimit of K-projective complexes of projective modules (cf. [8], [12, Remark 3.9]).

Proposition 4.1. *If X is a K-flat complex of flat modules and Y an acyclic complex of cotorsion modules, then any cochain map $X \longrightarrow Y$ is null-homotopic, i.e. $Hom_{\mathbf{K}(R)}(X, Y) = 0$.*

Proof. As noted above, there exists a filtered direct system of complexes $(X_i)_i$, such that X_i is a K-projective complex of projective modules for all i and $X \simeq \mathrm{colim}_i X_i$ in the homotopy category $\mathbf{K}(R)$. Since Y is acyclic (and the same is, of course, true for all of its translates), we have $\mathrm{Hom}_{\mathbf{K}(R)}(X_i, S^n Y) = 0$ for all i and all $n \in \mathbb{Z}$. We can now apply Theorem 2.1, by letting $\mathfrak{X}$ be the class of flat modules therein. Then, all of the hypotheses made at the beginning of §2 are satisfied: (b) follows since projective modules are flat and (c) expresses a well-known property of flatness. Finally, hypothesis (d) holds since any pure monomorphism $\iota : M \longrightarrow M'$ of flat modules has a flat cokernel, so that $\mathrm{Ext}^1_R(\mathrm{coker}\,\iota, Y^n) = 0$, in view of the fact that Y^n is a cotorsion module for all n. Therefore, we may conclude that $\mathrm{Hom}_{\mathbf{K}(R)}(X, S^n Y) = 0$ for all $n \in \mathbb{Z}$; in particular, $\mathrm{Hom}_{\mathbf{K}(R)}(X, Y) = 0$, as needed. □

The special case of Proposition 4.1, where the complex X therein is a flat module supported in degree 0, recovers a result by Bazzoni et al. [5], as we shall now explain.

Lemma 4.2. *Let* $0 \longrightarrow M' \longrightarrow M \xrightarrow{p} M'' \longrightarrow 0$ *be a short exact sequence of modules.*
(i) If M' is cotorsion, then the additive map $p_* : Hom_R(F,M) \longrightarrow Hom_R(F,M'')$, which is induced by p, is surjective for any flat module F.
(ii) If M is cotorsion and the additive map $p_* : Hom_R(F,M) \longrightarrow Hom_R(F,M'')$, which is induced by p, is surjective for any flat module F, then M' is cotorsion.

Proof. Both claims follow from the exact sequence

$$\mathrm{Hom}_R(F,M) \xrightarrow{p_*} \mathrm{Hom}_R(F,M'') \longrightarrow \mathrm{Ext}^1_R(F,M') \longrightarrow \mathrm{Ext}^1_R(F,M),$$

where F is any (flat) module. □

Corollary 4.3. *Let Y be an acyclic complex of cotorsion modules.*
(i) The complex $Hom_R(F,Y)$ is acyclic for any flat module F.
(ii) (cf. [5, Theorem 4.1(2)]) The cosyzygy modules of Y are cotorsion.

Proof. (i) Let F be a flat module, which is regarded as a cochain complex concentrated in degree 0; this complex is K-flat. Then, Proposition 4.1 implies that $\mathrm{Hom}_{\mathbf{K}(R)}(F,S^nY) = 0$, i.e. the n-th cohomology group of the cochain complex $\mathrm{Hom}_R(F,Y)$ is trivial for all n.

(ii) Let $Y = ((Y^n)_n, \partial)$ and consider its cosyzygy modules Z^n, $n \in \mathbb{Z}$. Then, for any $n \in \mathbb{Z}$ there is a short exact sequence of modules

$$0 \longrightarrow Z^n \longrightarrow Y^n \xrightarrow{\theta} Z^{n+1} \longrightarrow 0,$$

where θ is induced by the differential $\partial : Y^n \longrightarrow Y^{n+1}$. Since the module Y^n is cotorsion, in order to show that Z^n is also cotorsion, it suffices to show that the additive map

$$\theta_* : \mathrm{Hom}_R(F,Y^n) \longrightarrow \mathrm{Hom}_R(F,Z^{n+1}),$$

which is induced by θ, is surjective for all n and all flat modules F (cf. Lemma 4.2(ii)). Let us fix a flat module F and an integer n. Then, the linear maps $F \longrightarrow Z^{n+1}$ are precisely the $(n+1)$-cocycles of the complex $\mathrm{Hom}_R(F,Y)$ and those linear maps $F \longrightarrow Z^{n+1}$ that factor through $\theta : Y^n \longrightarrow Z^{n+1}$ are its $(n+1)$-coboundaries. Hence, the surjectivity of θ_* follows from the fact that the $(n+1)$-th cohomology group of the complex $\mathrm{Hom}_R(F,Y)$ is trivial; cf. (i) above. □

Remarks 4.4. (i) Stovicek [26] has proved that any pure acyclic complex X is left orthogonal to the class $\mathbf{K}(\mathrm{PInj})$ of all complexes of pure injective modules. If X is a pure acyclic complex of flat modules, then Bazzoni et al. [5] proved that X is actually left orthogonal to the bigger class $\mathbf{K}(\mathrm{Cotor})$ of all complexes of cotorsion modules. There is a similar picture for K-flat complexes. Proposition 3.1 asserts that any K-flat complex X is left orthogonal to the class $\mathbf{K}_{ac}(\mathrm{PInj})$ of all acyclic complexes of pure injective modules. If X is a K-flat complex of flat modules, then Proposition 4.1 asserts that X is actually left orthogonal to the bigger class $\mathbf{K}_{ac}(\mathrm{Cotor})$ of all acyclic complexes of cotorsion modules. The inclusion $\mathbf{K}_{ac}(\mathrm{PInj}) \subseteq \mathbf{K}_{ac}(\mathrm{Cotor})$ induces a reverse inclusion between the left orthogonals ${}^{\perp}\mathbf{K}_{ac}(\mathrm{PInj}) \supseteq {}^{\perp}\mathbf{K}_{ac}(\mathrm{Cotor})$, which becomes an equality

$$\mathbf{K}(\mathrm{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\mathrm{Pinj}) = \mathbf{K}(\mathrm{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\mathrm{Cotor}) \tag{4}$$

when restricted to complexes of flat modules. Both sides of the latter equality coincide with the class of K-flat complexes of flat modules, in view of the chain of inclusions

$$\mathbf{K}(\mathrm{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\mathrm{Cotor}) \subseteq \mathbf{K}(\mathrm{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\mathrm{PInj}) = \mathbf{K}(\mathrm{Flat}) \cap \mathbf{K}\text{-flat} \subseteq \mathbf{K}(\mathrm{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\mathrm{Cotor}).$$

(ii) We have demonstrated above how one can deduce Corollary 4.3(ii), using Proposition 4.1. It should be noted though that results already available in the literature can be used in order to show that Corollary 4.3(ii) actually implies the (apparently stronger) result in Proposition 4.1. Indeed, Gillespie has introduced in [14] a method for constructing cotorsion pairs in the category of complexes out of cortorion pairs in the category of modules. Applying his method to the cotorsion pair (Flat,Cotor) in the module category, and using [14, Proposition 4.11] and Corollary 4.3(ii), one can deduce the claim made in Proposition 4.1. To be more precise, using Gillespie's terminology, Corollary 4.3(ii) implies that the class of all acyclic complexes of cotorsion modules coincides with the class $\widetilde{\text{Cotor}}$ of Cotor-complexes, whereas [14, Proposition 4.11] asserts that the class of K-flat complexes of flat modules coincides with the class $dg\widetilde{\text{Flat}}$. As the latter class is defined by Gillespie to be the class of those complexes of flat modules which are left Hom-orthogonal to all complexes in $\widetilde{\text{Cotor}}$, we conclude that

$$\mathbf{K}(\text{Flat}) \cap \mathbf{K}\text{-flat} = \mathbf{K}(\text{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\text{Cotor})$$

(cf. (i) above). In particular, it follows that $\mathbf{K}(\text{Flat}) \cap \mathbf{K}\text{-flat} \subseteq {}^{\perp}\mathbf{K}_{ac}(\text{Cotor})$, which is precisely the assertion made in Proposition 4.1. We also note that [14, Corollary 4.18] implies that for any complex Z there exist short exact (approximation) sequences of complexes

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Z \longrightarrow Y' \longrightarrow X' \longrightarrow 0, \tag{5}$$

where X, X' are K-flat complexes of flat modules and Y, Y' are acyclic complexes of cotorsion modules.

(iii) In [7, Theorem 3.3] the authors proved the analogues of Theorem 4.1 and Corollary 4.3(ii) for complexes of quasi-coherent sheaves on any separated quasi-compact scheme.

Let Z be a complex of flat modules and consider the first of the two approximation sequences (5) above, namely the short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0,$$

where X is a K-flat complex of flat modules and Y is an acyclic complex of cotorsion modules. Since both X, Z are complexes of flat modules, it follows that Y is also a complex of flat modules, i.e. Y is an acyclic complex of flat cotorsion modules. Moreover, the short exact sequence is degree-wise split; hence, X is the mapping cone of a cochain map $\phi : S^{-1}Z \longrightarrow Y$ and the short exact sequence represents a distinguished triangle in the homotopy category $\mathbf{K}(\text{Flat})$ of flat modules

$$X \longrightarrow Z \xrightarrow{Sf} SY \longrightarrow SX,$$

where f is the homotopy class of ϕ. Let $\mathbf{K}_{ac}(\text{Flat-Cotor})$ be the triangulated subcategory of the homotopy category, consisting of those complexes which are homotopy equivalent to an acyclic complex of flat cotorsion modules. Since $\mathbf{K}_{ac}(\text{Flat-Cotor}) \subseteq \mathbf{K}_{ac}(\text{Cotor})$ is contained in the right Hom-orthogonal of the thick subcategory $\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat})$,[1] the existence of the approximation triangle above for any complex of flat modules Z, shows that

$$(\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}), \mathbf{K}_{ac}(\text{Flat-Cotor}))$$

[1] The triangulated subcategory $\mathbf{K}$-flat $\cap\, \mathbf{K}(\text{Flat})$ is closed under (countable) coproducts in $\mathbf{K}(R)$; hence, it is thick in the full homotopy category $\mathbf{K}(R)$.

is a Bousfield localizing pair in $\mathbf{K}$(Flat). The argument in [23, Remark 9.1.15] shows that the pair is orthogonal in $\mathbf{K}$(Flat), i.e.

$$\mathbf{K}(\text{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\text{Flat-Cotor})) = \mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}) \tag{6}$$

and

$$\mathbf{K}(\text{Flat}) \cap (\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat}))^{\perp} = \mathbf{K}_{ac}(\text{Flat-Cotor}).$$

It follows that $\mathbf{K}_{ac}$(Flat-Cotor) is a thick subcategory of the homotopy category $\mathbf{K}$(Flat); if $X, Y \in \mathbf{K}$(Flat) and $X \oplus Y \in \mathbf{K}_{ac}$(Flat-Cotor), then $X, Y \in \mathbf{K}_{ac}$(Flat-Cotor). We record the discussion above in the form of the following result; assertions (i), (ii) and (iii) follow formally from general principles concerning Bousfield localization (cf. [23, Chapter 9], [18, Proposition 4.9.1]).

Theorem 4.5. *The pair* ($\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*), $\mathbf{K}_{ac}$(*Flat-Cotor*)) *is a Bousfield localizing pair in the homotopy category* $\mathbf{K}$(*Flat*) *of flat modules. Consequently, the following assertions hold:*

(i) The inclusion functor $\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*) $\hookrightarrow$ $\mathbf{K}$(*Flat*) *admits a right adjoint and the inclusion functor* $\mathbf{K}_{ac}$(*Flat-Cotor*) $\hookrightarrow$ $\mathbf{K}$(*Flat*) *admits a left adjoint.*

(ii) Let $\jmath_\rho : \mathbf{K}(\mathit{Flat}) \longrightarrow \mathbf{K}\text{-}\mathit{flat} \cap \mathbf{K}(\mathit{Flat})$ *be a right adjoint functor to the inclusion* $\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*) $\hookrightarrow$ $\mathbf{K}$(*Flat*). *Then, for any complex of flat modules* Z *the counit of adjunction morphism* $\jmath_\rho Z \longrightarrow Z$ *is a* $\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*) *precover.*

(iii) Let $\mathbf{K}$(*Flat*)/($\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*)) *be the Verdier quotient of* $\mathbf{K}$(*Flat*) *by the triangulated subcategory* $\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*). *Then, the quotient functor*

$$q : \mathbf{K}(\mathit{Flat}) \longrightarrow \mathbf{K}(\mathit{Flat})/(\mathbf{K}\text{-}\mathit{flat} \cap \mathbf{K}(\mathit{Flat}))$$

admits a right adjoint and the composition

$$\mathbf{K}_{ac}(\mathit{Flat\text{-}Cotor}) \hookrightarrow \mathbf{K}(\mathit{Flat}) \xrightarrow{q} \mathbf{K}(\mathit{Flat})/(\mathbf{K}\text{-}\mathit{flat} \cap \mathbf{K}(\mathit{Flat}))$$

is an equivalence of categories. In particular, the quotient $\mathbf{K}$(*Flat*)/($\mathbf{K}$-*flat* $\cap$ $\mathbf{K}$(*Flat*)) *has small Hom-sets.* □

If we identify the Verdier quotient $\mathbf{K}$(Flat)/($\mathbf{K}$-flat $\cap$ $\mathbf{K}$(Flat)) with $\mathbf{K}_{ac}$(Flat-Cotor), by means of the composition

$$\mathbf{K}_{ac}(\text{Flat-Cotor}) \hookrightarrow \mathbf{K}(\text{Flat}) \xrightarrow{q} \mathbf{K}(\text{Flat})/(\mathbf{K}\text{-flat} \cap \mathbf{K}(\text{Flat})),$$

then the right adjoint to the quotient functor q is identified with the inclusion functor

$$\mathbf{K}_{ac}(\text{Flat-Cotor}) \hookrightarrow \mathbf{K}(\text{Flat})$$

and, of course, the left adjoint to the latter inclusion functor is identified with the quotient functor q.

Remark 4.6. As we noted in Remark 4.4(i), both sides of equation (4) are equal to the class $\mathbf{K}$(Flat) $\cap$ $\mathbf{K}$-flat of K-flat complexes of flat modules. Therefore, equation (6) above shows that

$$\mathbf{K}(\text{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\text{Flat-Cotor})) = \mathbf{K}(\text{Flat}) \cap {}^{\perp}\mathbf{K}_{ac}(\text{Cotor}).$$

In other words, the inclusion $\mathbf{K}_{ac}$(Flat-Cotor) $\subseteq$ $\mathbf{K}_{ac}$(Cotor) induces a reverse inclusion between the left orthogonals ${}^{\perp}\mathbf{K}_{ac}$(Flat-Cotor) $\supseteq$ ${}^{\perp}\mathbf{K}_{ac}$(Cotor), which becomes an equality when restricted to complexes of flat modules.

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